Analytic Number Theory
Instructor: Ranjan Bera
M. Math End Semester Exam(2022).
Maximum marks: 50

Time: 3 Hours

Answer the following questions.

1. Find all integers n such that $\phi(n) = 16$. Prove that

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}$$

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2+3

2. Find the Dirichlet characters for $k = 9$. Give an example of a Dirichlet character f mod k for which k is not the smallest positive period of f.

3+2.

3. Define primitive character mod k. Prove that there exists no real primitive character χ mod k if $k = 2m$, where m is odd.

2+3

4. Prove that the conductor of χ divides every induced modulus for χ. If n and k are integers, $n > 0$, let $G(k; n) = \sum_{r=1}^{n} e^{2\pi ikr^2/n}$, prove that $G(k; mn) = G(km; n)G(kn; m)$ whenever $(m, n) = 1$.

3+2

5. State and prove law of Quadratic reciprocity, for odd primes.

1+4

6. If P is an odd positive integer, Prove that

$$(-1|P) = (-1)^{(P-1)/2}$$

and

$$(2|P) = (-1)^{(P^2-1)/8}.$$

Determine whether -104 is a quadratic residue or non residue of the prime 997.

2+2+1.

7. Let g be a primitive root mod p, where p is an odd prime, then prove that either g or $g + p$ is a primitive root mod p^e for all $e \geq 2$.

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8. Show that the zeros of Riemann's Xi-function $\xi(s)$ (if any exist) are all situated in the strip $0 \leq \Re(s) \leq 1$ and lie symmetrically about the lines $t = 0$ and $\Re(s) = 1/2$.

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9. Prove that the Diophantine equation

$$y^2 = x^3 + k$$

has no solutions if k has the form

$$k = (4n - 1)^3 - 4m^2$$

where m and n are integers such that no prime $p \equiv -1 \pmod 4$ divides m

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10. Prove that the Bernoulli polynomials satisfy the relations $B_n(1 - x) = (-1)^n B_n(x)$ and $B_{2n+1}(1/2) = 0$ for every $n \geq 0$.

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